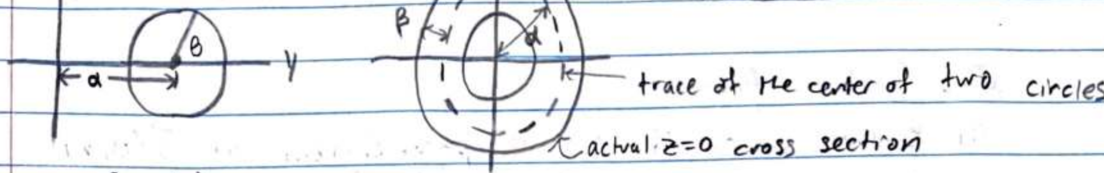
11/22 Surfaces and Calculus

Recall: A surface is $\vec{S}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ on domain D.

Ex: the torus w/ major radius α and minor radius β (for $\alpha > \beta > 0$) is the surface w/ equation

$$\vec{S}(u,v) = \langle (\alpha + \beta\cos(u))\cos(v),\ (\alpha+\beta\cos(u))\sin(v),\ \sin(u) \rangle$$

on domain $D = [0, 2\pi] \times [0, 2\pi]$ "mathy donut"



$r = \alpha + \beta\cos(\theta)$

I. Tangent Planes

The tangent plane to surface $\vec{S}(u,v)$ at input point (a,b) has normal vector $\vec{n}(a,b) = \vec{S}_u(a,b) \times \vec{S}_v(a,b)$ where

$\vec{S}_u = \frac{\partial \vec{S}}{\partial u} = \langle x_u, y_u, z_u \rangle$ ← vector of partial derivatives

Ex: Compute the tangent plane to the torus w/ major radius 4 and minor radius 1 at point $\vec{S}(3\pi/4, \pi/4)$

Sol: we want $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$ and we're given $\vec{p} = \vec{S}(\frac{3\pi}{4}, \frac{\pi}{4})$

$\vec{S}(u,v) = \langle (4+\cos(u))\cos(v), (4+\cos(u))\sin(v), \sin(u) \rangle$ on $[0, 2\pi]^2$

$\therefore \vec{p} = \vec{S}(\frac{3\pi}{4}, \frac{\pi}{4}) = \langle (4+\cos(\frac{3\pi}{4}))\cos(\frac{\pi}{4}), (4+\cos(\frac{3\pi}{4}))\sin(\frac{\pi}{4}), \sin(\frac{3\pi}{4}) \rangle$

$= \langle (4 - 1/\sqrt{2})1/\sqrt{2}, (4-1/\sqrt{2})1/\sqrt{2}, 1/\sqrt{2} \rangle = \langle 4/\sqrt{2} - 1/2, 4/\sqrt{2} - 1/2, 1/\sqrt{2} \rangle$

(Comment: we just need normal at $(3\pi/4, \pi/4)$, but we'll compute it more generally for use later on)

$\vec{n} = \vec{S}_u \times \vec{S}_v$

$\vec{S}_u = \langle -\sin(u)\cos(v), -\sin(u)\sin(v), \cos(u) \rangle$ $\quad \vec{S}_v = \langle -\sin(v)(4+\cos(u)), \cos(v)(4+\cos(u)), 0 \rangle$

$$\vec{n}(u,v) = \vec{S}_u(u,v) \times \vec{S}_v(u,v) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(u)\cos(v) & -\sin(u)\sin(v) & \cos(u) \\ -\sin(v)(4+\cos(u)) & \cos(v)(4+\cos(u)) & 0 \end{vmatrix}$$

$$= (0 - \cos(u)\cos(v)(4+\cos(u)))\vec{i} - (0 - \cos(u)(-\sin(v)(4+\cos(u))))\vec{j}$$
$$+ (-\sin(u)\cos(v)\cos(v)(4+\cos(u)) - \sin(u)\sin(v)\sin(v)(4+\cos(u)))\vec{k}$$

$$= -(4+\cos(u)) \langle \cos(u)\cos(v), \cos(u)\sin(v), \sin(u) \rangle$$

(normal vector to the torus at every input point (u,v)

$\therefore \vec{n}\left(\frac{3\pi}{4}, \frac{\pi}{4}\right) = -(4+\cos(\frac{3\pi}{4}))\langle \cos(\frac{3\pi}{4})\cos(\frac{\pi}{4}), \cos(\frac{3\pi}{4})\sin(\frac{\pi}{4}), \sin(\frac{3\pi}{4})\rangle$

$= -(4-1/\sqrt{2})\langle -1/\sqrt{2}\cdot 1/\sqrt{2}, -1/\sqrt{2}\cdot 1/\sqrt{2}, 1/\sqrt{2}\rangle$

$= -(4-1/\sqrt{2})\langle -1/2, -1/2, 1/\sqrt{2}\rangle$

$\therefore$ tangent plane to this torus at $\vec{s}(\frac{3\pi}{4}, \frac{\pi}{4})$ is

$\vec{n}\cdot(\vec{x}-\vec{p})=0$ i.e. $\vec{n}(\frac{3\pi}{4}, \frac{\pi}{4})\cdot(\vec{x}-\vec{s}(\frac{3\pi}{4}, \frac{\pi}{4}))=0$

i.e. $-(4-1/\sqrt{2})\langle -1/2, -1/2, 1/\sqrt{2}\rangle \cdot \langle x-4/\sqrt{2}+1/2, y-4/\sqrt{2}+1/2, z-1/\sqrt{2}\rangle = 0$

i.e. $-1/2(x-4/\sqrt{2}+1/2) - 1/2(y-4/\sqrt{2}+1/2) + 1/\sqrt{2}(z-1/\sqrt{2}) = 0$

II Surface Area

The surface area of surface parameterized by $\vec{s}(u,v)$ on domain D is $\text{Area}(S) = \iint_D |\vec{s}_u \times \vec{s}_v| \, dA$

Why that formula? Piecewise-linearly approx. surface S via parallelograms. Limit sums of those area approximations (See website for geogebra approximations)

Ex: for the torus w/ major radius 4 and minor radius 1, compute the surface area

Sol: $\text{Area}(S) = \iint_D |\vec{s}_u \times \vec{s}_v| \, dA$

from before $\vec{s}_u(u,v) \times \vec{s}_v(u,v) = -(4+\cos(u))\langle \cos(u)\cos(v), \cos(u)\sin(v), \sin(u)\rangle$

So we compute

$|\vec{s}_u \times \vec{s}_v| = |-(4+\cos(u))| \sqrt{\cos^2(u)\cos^2(v) + \cos^2(u)\sin^2(v) + \sin^2(u)}$

$= |4+\cos(u)| \sqrt{\cos^2(u)(\cos^2(v)+\sin^2(v)) + \sin^2(u)}$

$= 4+\cos(u)$

$\int_{u=0}^{2\pi}\int_{v=0}^{2\pi} 4+\cos(u) \, dv \, du$

$\int_{u=0}^{2\pi} (4+\cos(u)) [v]_0^{2\pi} \implies 16\pi^2$

Exercise: Compute surface area of general torus w/ major radius α and minor radius β $(\alpha > \beta > 0)$

III - Surface Integral

The (surface) integral of function $f(x,y,z)$ over surface S parameterized by $\vec{S}(u,v)$ on domain D is

$$\iint_S f \, dS = \iint_D f(\vec{S}(u,v)) \, |\vec{S}_u \times \vec{S}_v| \, dA$$

Note: the correct way to understand "$dS = |\vec{S}_u \times \vec{S}_v| \, dA$" is via a Jacobian.

$|\vec{S}_u \times \vec{S}_v|$ is the Jacobian of a coordinate change